UNIT-2

(i) Classification of Linear P.D.E. of 2nd order
(ii) Method of separation of variable
(iii) Solution of Wave and Heat Conduction equation in two dimension
(iv) Laplace equation in two dimension
(v) Equation of Transmission Lines

Applications of Partial differential equation

Type-1. Method of Separation of Variable:-

Problem① Solve by method of separation of variable:-

$$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

Solve:- Given that

$$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \quad \text{---} ①$$

Let $u(x,y) = X(x)\,Y(y)$ ——— ②

be solution of ①

$$\therefore \frac{\partial u}{\partial x} = Y\frac{\partial X}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

and $$\frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

Therefore ① becomes

$$2Y\frac{\partial^2 X}{\partial x^2} - X\frac{\partial Y}{\partial y} = 0$$

Divide by XY; we get,

$$\frac{2}{X}\frac{\partial^2 X}{\partial x^2} - \frac{1}{Y}\frac{\partial Y}{\partial y} = 0$$

$$\frac{2}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial Y}{\partial y} = k \quad (\text{Say})$$

Consider, $\frac{2}{X}\frac{\partial^2 X}{\partial x^2} = k$

$$\frac{\partial^2 X}{\partial x^2} - \frac{k}{2}X = 0$$

Auxiliary equation –

$$m^2 - \frac{k}{2} = 0$$

$$m = \pm\sqrt{\frac{k}{2}}$$

Hence solution is

$$X = c_1 e^{\sqrt{k/2}\,x} + c_2 e^{-\sqrt{k/2}\,x} \quad \text{—— (3)}$$

Now consider,

$$\frac{1}{Y}\frac{dY}{dy} = k$$

$$\frac{dY}{\cancel{dy}\,Y} = k\,dy$$

$$\log Y = ky + \log c_3$$

$$\log \frac{Y}{c_3} = ky$$

$$\frac{Y}{c_3} = e^{ky}$$

$$Y = c_3 e^{ky} \quad \text{—— (4)}$$

from (3) and (4)

The solution (2) becomes

$$u(x,y) = \left(c_1 e^{\sqrt{k/2}\,x} + c_2 e^{-\sqrt{k/2}\,x}\right) c_3 e^{ky}$$

$$\boxed{u(x,y) = \left(A e^{\sqrt{k/2}\,x} + B e^{-\sqrt{k/2}\,x}\right) e^{ky}}$$

Problem-2 Solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{--- (1)} \quad \text{when } u(x,0) = 6e^{-3x}.$$

Solve :- Let, $u(x,t) = X(x)\,T(t)$ --- (2)

be solution of (1)

$$\frac{\partial u}{\partial x} = T\frac{\partial X}{\partial x}$$

$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

Therefore (1) becomes

$$T\frac{\partial X}{\partial x} = 2X\frac{\partial T}{\partial t} + XT$$

Divide by XT; we get,

$$\frac{1}{X}\frac{\partial X}{\partial x} = \frac{2}{T}\frac{\partial T}{\partial t} + 1 = k \text{ (say)} \quad \text{--- (3)}$$

Consider $\frac{1}{X}\frac{\partial X}{\partial x} = k$

$$\frac{\partial X}{X} = k\,\partial x$$

Integrate, $\log X = kx + \log c_1$

$$\log\frac{X}{c_1} = kx$$

$$\frac{X}{c_1} = e^{kx}$$

$$X = c_1 e^{kx} \quad \text{--- (4)}$$

Now, from (3); consider

$$\frac{2}{T}\frac{\partial T}{\partial t} + 1 = k$$

$$\frac{\partial T}{T} = \left(\frac{k-1}{2}\right)\partial t$$

$$\log T = \left(\frac{k-1}{2}\right)t + \log c_2$$

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$u_x = \frac{\partial u}{\partial x}$ $\quad u_z = \frac{\partial u}{\partial z}$

$u_y = \frac{\partial u}{\partial y}$

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$$\log \frac{T}{c_2} = \left(\frac{k-1}{2}\right)t$$

$$\frac{T}{c_2} = e^{\left(\frac{k-1}{2}\right)t}$$

$$T = c_2 e^{\left(\frac{k-1}{2}\right)t} \quad \text{——} \ (5)$$

From (4) and (5); we get;

the solution (2) becomes

$$u(x,t) = c_1 e^{kx} \, c_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$u(x,t) = c e^{kx} \cdot e^{\left(\frac{k-1}{2}\right)t} \quad \text{——} \ (6)$$

$$u(x,0) = c e^{kx} \cdot e^{\left(\frac{k-1}{2}\right)\cdot 0}$$

$$u(x,0) = c e^{kx}$$

$$6e^{-3x} = c e^{kx}$$

$$\boxed{c = 6}$$

& $\boxed{k = -3}$

Therefore, (6) becomes

$$\boxed{u(x,y) = 6e^{-(3x+2t)}}$$

Problem (3) :- Solve $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$

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Classification of Partial differential equation :-

Consider 2^{nd} order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y^2} + C\frac{\partial^2 u}{\partial z^2} + f\left(x, y, z, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0 \quad \text{--- (1)}$$

where A = positive

Therefore (1) represent

Elliptic if $B^2 - 4AC < 0$

Parabolic if $B^2 - 4AC = 0$

Hyperbolic if $B^2 - 4AC > 0$

Remark:- (1) If A, B, C are constant

Then nature of (1) will same in whole region i.e. for all x and y.

(2) If A, B, C are function of x and y then nature of (1) depend on x and y.

Problem :- (1):- Classify the P.D.E.

(i) $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2}$

(ii) $\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$

(iii) $\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2}$

Solve:- (i) $A = 1$

$B = 1$

$C = 1$

$\therefore B^2 - 4AC = 1 - 4 = -3 < 0$

$\therefore$ It will represent "Elliptic".

(ii) $A = 1$, $B = -4$, $C = 1$

$B^2 - 4AC = 1 + 16 = 17 > 0$

$\therefore$ It will represent "Hyperbolic".

(iii) $A=1,\ B=4,\ C=4$

$B^2-4AC = 16-16=0$

$\therefore$ It will represent "Parabolic".

Problem :- ② Find the nature of following

(i) $x^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0$

(ii) $t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$

Solve :- (i) $A=x^2,\ B=0,\ C=-1$

$B^2-4AC = 0+4x^2 = 4x^2 \geq 0$

(a) $x\neq 0$, It will represent "Hyperbolic".

(b) $x=0$ It will represent "Parabolic".

Some important PDE :-

① One dimensional wave equation :-

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ where $c^2 =$ constant

② One dimensional Heat equation :-

$$\boxed{\frac{\partial^2 u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

③ Laplace equation (Two dimensional heat flow) :-

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

④ Three dimensional laplace equation :-

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0}$$

One dimensional wave equation :-

① Nature $\quad c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \left\{ \begin{array}{l} x \rightarrow \text{displacement} \\ t \rightarrow \text{time} \end{array} \right\}$

$A = c^2, \ B = 0, \ c = -1$

$\therefore \ B^2 - 4AC = 4c^2 \geq 0$

$\therefore$ It will represent hyperbolic.

② Solution

$\rightarrow$ D'Alembert Solution

Case (I) $\quad u(x, 0) = f(x)$

$u_t(x, 0) = 0$ i.e. $\left. \frac{\partial u}{\partial t} \right|_{t=0} = 0$

Then solution of ①

$$\boxed{u(x,t) = \frac{1}{2} \left[f(x+ct) + f(x-ct) \right]}$$

<u>Case-(II)</u> $u(x,0) = f(x)$

$u_t(x,0) = g(x)$ i.e. $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

Then solution is

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} g(s)\, ds$$