Type-3. When $f(x,y) = x^m y^n$

Note:-

$D = \frac{\partial}{\partial x}$ $\quad \frac{1}{D} = \int$

$D' = \frac{\partial}{\partial y}$ $\quad \frac{1}{D'} = \int$

Case:-(i) When $m > n$

Then $PI = \frac{1}{f(D,D')} Q(x,y)$

$= [f(D,D')]^{-1} Q(x,y)$

expand $f(D,D')$ in terms of $\frac{D'}{D}$

Binomial expansion

Case:-(ii) When $m < n$

Expand $f(D,D')$ in terms of $\frac{D}{D'}$

Problem:- $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 3x^2 y.$

Solve:- $(D^3 - 2D^2D')z = 3x^2y$

Auxiliary equation

$m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

$m = 0, 0, 2$

$C.F. = f_1(y) + x f_2(y) + f_3(y+2x)$

$P.I. = \frac{1}{D^3 - 2D^2D'} \cdot 3x^2y$

$= \frac{1}{D^3}\left[1 - \frac{2D'}{D}\right]^{-1} 3x^2y$

$= \frac{1}{D^3}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \cdots\cdots\right] 3x^2y$

$= \frac{1}{D^3}\left[3x^2y + \frac{2}{D}(D'3x^2y) + \frac{4}{D^2}(D'^2 3x^2y) + \cdots\right]$

$$= \frac{1}{D^3}\left[3x^2y + \frac{2}{D}3x^2 + \frac{4}{D^2}(0) + 0 - \cdots\right]$$

$$= \frac{1}{D^3}\left[3x^2y + 6 \cdot \frac{1}{D}(x^2)\right]$$

$$= \frac{1}{D^3}\left[3x^2y + 6 \cdot \frac{x^3}{3}\right] \qquad \left[\because \frac{1}{D}(x^2) = \int x^2 dx\right]$$

$$= \frac{1}{D^3}\left[3x^2y + 2x^3\right]$$

$$= \frac{1}{D^2}\left[x^3y + \frac{x^4}{2}\right]$$

$$= \frac{1}{D}\left[\frac{x^4y}{4} + \frac{x^5}{10}\right]$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$$P.I. = \frac{x^5y}{20} + \frac{x^6}{60}$$

Hence solution

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y) + x f_2(y) + f_3(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60}}$$ Ans.

Problem :- $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$

Solve :- $(D^2 - D'^2)z = x - y$

Auxiliary equation

$m^2 - 1 = 0$

$m = \pm 1$

$C.F. = f_1(y+x) + f_2(y-x)$

$$P.I. = \frac{1}{(D^2 - D'^2)}(x - y)$$

$$= \frac{1}{D^2}\left[1-\left(\frac{D'}{D}\right)^2\right]^{-1}(x-y)$$

$$= \frac{1}{D^2}\left[1+\frac{D'^2}{D^2}+\frac{D'^4}{D^4}+\ldots\ldots\right](x-y)$$

$$= \frac{1}{D^2}\left[(x-y)+\frac{1}{D^2}D'^2(x-y)+\ldots\ldots\right]$$

$$= \frac{1}{D^2}\left[(x-y)+0+\ldots\ldots\right]$$

$$= \frac{1}{D}\int(x-y)\,dx$$

$$= \frac{1}{D}\left[\frac{x^2}{2}-xy\right]$$

$$= \int\left(\frac{x^2}{2}-xy\right)dx$$

$$P.I. = \frac{x^3}{6}-\frac{x^2y}{2}$$

Hence solution

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y+x)+f_2(y-x)+\frac{x^3}{6}-\frac{x^2y}{2}}$$ Ans.

Problem :- $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2} = x^2+y^2$.

Solve :-

Formula :-

$$\frac{1}{D-mD'}f(x,y) = \int f(x, c-mx)\,dx$$

Formula :-

$$\frac{1}{D-mD'} Q(x,y) = \int Q(x, c-mx)\,dx$$

By putting $y = c-mx$

$$\frac{1}{D+mD'} Q(x,y) = \int Q(x, c+mx)\,dx$$

By putting $y = c+mx$

Problem :- $(D-D')(D+2D')z = (y+1)e^x$

Solve :- Auxiliary equation

$$(m-1)(m+2) = \cancel{(y+1)e^x}\ 0$$

$$m = 1, -2$$

$$\therefore C.F. = f_1(y+x) + f_2(y-2x)$$

$$P.I. = \frac{1}{(D-D')(D+2D')}(y+1)e^x$$

$$= \frac{1}{(D-D')}\left[\frac{1}{(D+2D')}(y+1)e^x\right]$$

$$= \frac{1}{(D-D')}\int \underbrace{(c+2x+1)}_{I}\underbrace{e^x}_{II}\,dx$$

By putting $y = c+2x$

$$= \frac{1}{(D-D')}\left[(c+2x+1)e^x - 2e^x\right]$$

$$= \frac{1}{(D-D')}\left[(c+2x)e^x - e^x\right]$$

$$= \frac{1}{(D-D')}\left[ye^x - e^x\right] \quad [\because y = c+2x]$$

$$= \frac{1}{(D-D')}(y-1)e^x$$

$$= \frac{1}{(D-D')}(y-1)e^x$$

$$= \int \underbrace{(c-x-1)}_{I}\,\underbrace{e^x}_{II}\,dx$$

By putting $y = c-x$

$$= (c-x-1)e^x + e^x$$

$$= (c-x)e^x$$

$$P.I. = ye^x$$

Hence, solution $z = C.F. + P.I.$

$$\boxed{z = f_1(y+x) + f_2(y-2x) + ye^x}$$

Problem :- ② $(D^2 - D'^2)z = \tan^3 x \cdot \tan y - \tan x \cdot \tan^3 y$

Solve :- $(D-D')(D+D')z = \tan^3 x \cdot \tan y - \tan x \cdot \tan^3 y$

Type :- When $Q(x,y) = e^{ax+by} V(x,y)$

$$\frac{1}{f(D,D')} e^{ax+by} V(x,y)$$

$$= e^{ax+by} \frac{1}{f(D+a, D'+b)} V(x,y)$$

By putting $D = D+a$

$D' = D'+b$

~~Problem~~ :-

Non-Homogeneous Partial Differential Equation :-

① Consider non-homogeneous P.D.E.

$$(D - mD' - \alpha) z = 0$$

Then

$$C.F. = e^{\alpha x} f(y+mx)$$

Ex:- (i) $(D-2D'-3) z = 0$

Solve :- $z = e^{3x}(y+2x)$

(ii) $(D-2D'+3) z = 0$

Solve :- $z = e^{-3x}(y+2x)$

(iii) $(D+2D'+3) z = 0$

Solve :- $z = e^{-3x}(y-2x)$

Formula - ② If $(D - mD' - \alpha)^2 z = 0$

Then solution

$$z = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx)$$

Formula - ③ If $(D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2) \dots\dots (D - m_nD' - \alpha_n) z = 0$

Then solution

$$z = e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x) + \dots\dots + e^{\alpha_n x} f_n(y+m_nx)$$

Ex:- $(D-2D'+3)(D-3D'-4)z=0$

Solve:- $z = e^{-3x} f_1(y+2x) + e^{4x} f_2(y+3x)$

Problem:- $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

Solve:- $C.F. = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x)$

$$P.I. = \frac{1}{(D-3D'-2)^2} 2e^{2x}\tan(y+3x)$$

$$= 2e^{2x} \frac{1}{(D+2-3(D'+0)-2)^2}\tan(y+3x)$$

$$= 2e^{2x} \frac{1}{(D-3D')^2}\tan(y+3x)$$

$$= 2e^{2x} \frac{1}{(D-3D')}\left[\frac{1}{(D-3D')}\tan(y+3x)\right]$$

$$= 2e^{2x} \frac{1}{(D-3D')}\int \cancel{e-3x} \tan(c-\cancel{3x}+\cancel{3x})\,dx$$

By putting $y = c-3x$

$$= 2e^{2x} \frac{1}{(D-3D')}\int \tan c\,dx$$

$$= 2e^{2x} \frac{1}{(D-3D')} x\tan c$$

$$= 2e^{2x} \frac{1}{(D-3D')} x\tan(y+3x)$$

$$= 2e^{2x} \int x\tan(c-\cancel{3x}+\cancel{3x})\,dx$$

$$= 2e^{2x}\int x\tan c\,dx$$

$$= \cancel{2}e^{2x}.\tan c.\frac{x^2}{\cancel{2}}$$

$$P.I. = e^{2x}.x^2\tan(y+3x)$$

Then solution $z = C.F. + P.I.$

$$\boxed{z = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + e^{2x}.x^2\tan(y+3x)}$$

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Case-II. For non-homogeneous :-
when $f(D, D')$ can't be factorized in linear form
$(D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2)\ldots\ldots$

Problem :- $(D^3-3DD'+D'+1)z=0$
Solve :- Solution

$$z=\sum Ae^{hx+ky}$$

where, h, k satisfying :-

$$(h^3-3hk+k+1)=0$$
$$(h^3+1)+k(1-3h)=0$$
$$k=\frac{h^3+1}{3h-1}$$

Hence, solution

$$z=\sum Ae^{hx+\left(\frac{h^3+1}{3h-1}\right)y}$$

Problem :- $(D-2D'-1)(D-2D'^2-1)z=0$
Solve :- C.F. corresponding to $(D-2D'-1)$
$=e^x f_1(y+2x)$

C.F. corresponding to $(D-2D'^2-1)$
$=\sum Ae^{hx+ky}$
where h, k satisfying second factor

$$(h-2k^2-1)=0$$
$$2k^2=h-1$$
$$k=\sqrt{\frac{h-1}{2}}$$
$$=\sum Ae^{hx+\sqrt{\frac{h-1}{2}}}$$

Hence, required solution

$$z=e^x f_1(y+2x)+\sum Ae^{hx+\sqrt{\frac{h-1}{2}}}$$

Ans

Problem :- $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = e^{2x+3y}$

Solve :- $(D^2 - D'^2 - 2D + 2D')z = e^{2x+3y}$

$[(D-D')(D+D') - 2(D-D')]z = e^{2x+3y}$

$[(D-D')(D+D'-2)]z = e^{2x+3y}$

$\therefore$ C.F. $= e^{0x} f_1(y+x) + e^{2x} f_2(y-x)$

$$P.I. = \frac{1}{(D-D')(D+D'-2)} e^{2x+3y}$$

$$P.I. = \frac{1}{(2-3)(2+3-2)} e^{2x+3y}$$

$$P.I. = -\frac{1}{3} e^{2x+3y}$$

Hence, solution is

$z =$ C.F. + P.I.

$$\boxed{z = f_1(y+x) + e^{2x} f_2(y-x) - \frac{1}{3} e^{2x+3y}}$$

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