Non-linear P.D.E. :-

Charpit's method :- Consider the non-linear P.D.E.

$$f(x, y, z, p, q) = 0$$

Now, Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

From these, we have the value of p and q

~~Si~~

Since,

$$dz = p\,dx + q\,dy$$

$z = z(x, y)$

Integrate, we get required solution

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy$$

Type-①. $\frac{dp}{0} = \frac{dq}{0} = \text{------}$

$dp = 0$

Integrate

$p = \text{Constant} = a$

and $dq = 0$

$q = \text{Constant}$

$q = b$

Since

$dz = p\,dx + q\,dy$

$dz = a\,dx + b\,dy$

Integrate

$$\boxed{z = ax + by + c}$$

Problem :- $z = px + qy + \underline{p^2 + q^2}$

Solve :- Let, $f = z - px - qy - p^2 - q^2$

$$\frac{\partial f}{\partial x} = -p$$

$$\frac{\partial f}{\partial y} = -q$$

$$\frac{\partial f}{\partial z} = 1$$

By Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \text{------}$$

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \text{------}$$

$$\frac{dp}{0} = \frac{dq}{0} = \text{------}$$

Now, Since

$$dp = 0$$

Integrate

$$\boxed{p = a}$$

Also, Since

$$dq = 0$$

Integrate

$$\boxed{q = b}$$

Since

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + q\,dy$$

Integrate

$$z = ax + by + c$$

$$\boxed{z = ax + by + (a^2 + b^2)}$$

Problem-② :- $px + qy = pq$ —①

Solve :- Let, $f = px + qy - pq$

$$\frac{\partial f}{\partial x} = p$$

$$\frac{\partial f}{\partial y} = q$$

$$\frac{\partial f}{\partial z} = 0$$

$$\frac{\partial f}{\partial p} = x - q$$

$$\frac{\partial f}{\partial q} = y - p$$

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrate

$$\log p = \log q + \log a$$

$$\log p = \log aq$$

$$\boxed{p = aq}$$

put in ①

$$aqx + qy = aq^2$$

$$ax + y = aq$$

$$q = \frac{ax + y}{a}$$

Hence $p = aq$

$$= a\left(\frac{ax+y}{a}\right)$$

$$p = ax + y$$

Since $dz = p\,dx + q\,dy$

$$\cancel{dz = aq\,dx + q\,dy}$$

$$\cancel{dz = q(a\,dx + dy)}$$

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

$$dz = \frac{(ax+y)}{a}(a\,dx + dy)$$

Integrate

$$z = \frac{1}{a}\int (ax+y)(a\,dx + dy) + b$$

put $ax + y = t$

$$a\,dx + dy = dt$$

$$\therefore z = \frac{1}{a}\int t\,dt + b$$

$$z = \frac{1}{a}\frac{t^2}{2} + b$$

$$\boxed{z = \frac{1}{a}\frac{(ax+y)^2}{2} + b}$$

Problem :- $zpq = p+q$

Solve :- Let, $f = p+q-zpq$

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0,\ \frac{\partial f}{\partial p} = 1-zq,\ \frac{\partial f}{\partial q} = 1-zp,\ \frac{\partial f}{\partial z} = -pq$$

$$\frac{\partial p}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{\partial q}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial z}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{\partial x}{-\frac{\partial f}{\partial p}} = \frac{\partial y}{-\frac{\partial f}{\partial q}}$$

$$\frac{\partial p}{-p^2q} = \frac{\partial q}{-q^2p}$$

$$\int\frac{\partial p}{p} = \int\frac{\partial q}{q}$$

$$\log p = \log q + \log c$$

$$\frac{p}{q} = c$$

$$\boxed{p = qc}$$

$$zpq = p+q$$

$$zq^2c = q(c+1)$$

$$q = \frac{c+1}{zc}$$

$$p = qc = \left(\frac{c+1}{z}\right)$$

$$\partial z = p\,\partial x + q\,\partial y$$

$$\partial z = \left(\frac{c+1}{z}\right)\partial x + \left(\frac{c+1}{zc}\right)\partial y$$

$$z\,\partial z = (c+1)\,\partial x + \frac{(c+1)}{c}\partial y$$

$$\frac{z^2}{2} = (c+1)x + \left(\frac{c+1}{c}\right)y + b$$

$$\frac{z^2}{2} = (c+1)\left[x + \frac{y}{c}\right] + b \text{ Ans}$$

Linear Partial Differential Equation with Constant Coefficient :-

Consider

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = Q(x, y) \quad \text{——— (1)}$$

Operator :- $D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

$$D^2 = \frac{\partial^2}{\partial x^2}, \quad D'^2 = \frac{\partial^2}{\partial y^2}$$

$$DD' = \frac{\partial^2}{\partial x \partial y}$$

$$D^2 D' = \frac{\partial^3}{\partial x^2 \partial y}, \quad DD'^2 = \frac{\partial^3}{\partial x \partial y^2}$$

Therefore (1) becomes,

$$(a_0 D^2 + a_1 DD' + a_2 D'^2) z = Q(x, y) \quad \text{——— (2)}$$

$$\boxed{f(D, D') z = Q(x, y)}$$

Auxiliary equation :-

Replace $D = m$, $D' = 1$

We get auxiliary equation

$$\boxed{a_0 m^2 + a_1 m + a_2 = 0}$$

It has two roots say m_1 and m_2

Case-I. When m_1 and m_2 are distinct

i.e. $m_1 \neq m_2$

Then

$$C.F. = f_1(y + m_1 x) + f_2(y + m_2 x)$$

or

C.F. = $f_1(x+m_1y) + f_2(x+m_2y)$

Case-② When $m_1 = m_2 = m$ (say)
Then
$$C.F. = f(y+mx) + xf(y+mx)$$
or
$$C.F. = f(x+my) + yf(x+my)$$

Particular Integral :-
$$P.I. = \frac{1}{f(D,D')} Q(x,y)$$

Problem :-

① $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$

② $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

③ $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$

Solve :- (i) $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$
$$(D^2 - a^2D'^2)z = 0$$
Auxiliary equation is
$$m^2 - a^2 = 0$$
$$\Rightarrow m = \pm a$$
Hence solution is
$$\boxed{z = f_1(y+ax) + f_2(y-ax)}$$

(ii) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

$(D^2 - DD' - 6D'^2)z = 0$

Auxiliary equation is

$m^2 - m - 6 = 0$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$(m-3)(m+2) = 0$

$m = -2, 3$

Hence solution is

$$\boxed{z = f_1(y-2x) + f_2(y+3x)}$$

(iii) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$

$(D^2 - 2DD' - D'^2)z = 0$

Auxiliary equation is :-

$m^2 - 2m - 1 = 0$

$m = \frac{2 \pm \sqrt{4 + 4 \times 1 \times 1}}{2}$

$= \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2}$

$m = 1 \pm \sqrt{2}$

$\therefore m_1 = 1 \pm \sqrt{2}$

$m_2 = 1 \pm \sqrt{2}$

Hence solution is

$$\boxed{z = f_1(y + (1+\sqrt{2})x) + f_2(y + (1-\sqrt{2})x)}$$

Problem :- $(D^4 - 6D^3D' + 9D^2D'^2)Z = 0$

Solve :- Auxiliary equation is :-

$$m^4 - 6m^3 + 9m^2 = 0$$
$$m^2(m^2 - 6m + 9) = 0$$
$$m^2(m^2 - 6m + 9) = 0$$
$$m^2(m-3)^2 = 0$$
$$m = 3, 3, 0, 0$$
$$m_1 = 0,\ m_2 = 0,\ m_3 = 3,\ m_4 = 3$$

Hence solution

$$\boxed{Z = f_1(y) + x f_2(y) + f_3(y+3x) + x f_4(y+3x)}$$

Consider

$$f(D, D')Z = Q(x, y)$$

Hence solution

$$\boxed{Z = C.F. + P.I.}$$

where $C.F.$ = solution of $f(D, D')Z = 0$

$$P.I. = \frac{1}{f(D, D')} Q(x, y)$$

Case-I. When $Q(x, y) = e^{ax+by}$

Then

$$\frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by} \ ; \ f(a,b) \neq 0$$

$$\left[\begin{array}{l} D \to a \\ D' \to b \end{array}\right]$$

if $f(a, b) = 0$

then $\dfrac{1}{f(D, D')} e^{ax+by} = x \dfrac{1}{\frac{\partial}{\partial D} f(D, D')} e^{ax+by}$

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Problem :- $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$

Solve :- $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Auxiliary equation is

$m^2 - 5m + 6 = 0$

$m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$(m-3)(m-2) = 0$

$m = 2, 3$

$\therefore$ C.F. $= f_1(y+2x) + f_2(y+3x)$

P.I. $= \frac{1}{D^2 - 5DD' + 6D'^2} e^{x+y}$

$= \frac{1}{(1-5+6)} e^{x+y}$ $[D \to 1, D' \to 1]$

$= \frac{1}{2} e^{x+y}$

Hence solution

$z =$ C.F. + P.I.

$$\boxed{z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2}e^{x+y}}$$

Problem :- ② $\frac{\partial^2 z}{\partial x^2} - \frac{4\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

Solve :- $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

Auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$\therefore$ C.F. $= f_1(y+2x) + xf_2(y+2x)$

$$P.I. = \frac{1}{(D^2 - 4DD' + 4D'^2)} e^{2x+y}$$

$$P.I. = \frac{1}{(4 - 4\times2\times1 + 4\times1)} e^{2x+y} \quad [D \to 2,\ D' \to 1]$$

$$P.I. = x \frac{1}{(2D - 4D')} e^{2x+y}$$

$$P.I. = x.x.\frac{1}{2} e^{2x+y}$$

$$P.I. = \frac{x^2}{2} e^{2x+y}$$

Hence solution

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}}$$

Type-②:- When $\phi(x, y) = \sin(ax+by)$ or $\cos(ax+by)$

$$\frac{1}{f(D^2, D'^2, DD')} \sin(ax+by) = \frac{1}{f(-a^2, -b^2, -ab)} \sin(ax+by)$$

when $f(-a^2, -b^2, -ab) \neq 0$

Problem:- $(D^3 - 4D^2D' + 4DD'^2) z = 6 \sin(3x+2y)$

Solve:- Auxiliary equation is

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m(m-2)^2 = 0$$

$$m = 0, 2, 2$$

Hence, $C.F. = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

$$P.I. = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 6\sin(3x+2y)$$

$$= 6.\frac{1}{D^2 D - 4D^2D' + 4DD'D'} \sin(3x+2y)$$

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$$= 6 \cdot \frac{1}{-9D + 36D' + 4D(-4)} \sin(3x+2y)$$

$$= 6 \cdot \frac{1}{-9D + 36D' - 16D} \sin(3x+2y)$$

$$= 6 \cdot \frac{1}{-25D + 36D'} \sin(3x+2y)$$

$$= 6 \cdot \frac{1}{(36D' - 25D)} \sin(3x+2y)$$

$$= 6 \cdot \frac{(36D' + 25D)}{(36D' - 25D)(36D' + 25D)} \sin(3x+2y)$$

$$= 6 \cdot \frac{(36D' + 25D)}{(36)^2 D'^2 - (25)^2 D^2} \sin(3x+2y)$$

$$= \frac{6}{(36)^2(-4) - (25)^2(-9)} (36D' + 25D) \sin(3x+2y)$$

$$= \frac{6}{441} \left[36 \times 2 \cos(3x+2y) + 25 \times 3 \cos(3x+2y) \right]$$

Hence, Solution is

$$z = C.F. + P.I.$$

$$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{6}{441} \left[72 \cos(3x+2y) + 75 \cos(3x+2y) \right]$$