UNIT - 1

Partial Differential Equation

An equation involving one dependent variable and its derivative w.r.t. one or more independent variable is called differential equation.

Differential equation

Ordinary differential equation (O.D.E.)	Partial differential equation (P.D.E.)
$\frac{dy}{dx} + y = 0$	$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$
$\frac{d^2y}{dx^2} + y = \sin x$	$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \sin(x+y)$

Partial Differential Equation (P.D.E.) :- An equation involving one dependent variable and its derivative w.r.t one or more than one independent variable is called P.D.E..

Notation :-

Let $z = f(x, y)$

Then $p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

Formation of Partial Differential equation (P.D.E.) :-

(i) Elimination of arbitrary constant

(ii) Elimination of arbitrary function

(i) Elimination of arbitrary constant :-

Let z be function of x and y connected by relation

$$f(x, y, z, a, b) = 0 \quad \text{--- (1)}$$

where a & b are constant

Differentiate partially w.r.t. x ~~and y~~, we get;

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$f = f(x, y)$

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \quad \text{--- (2)}$$

and $$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \quad \text{--- (3)}$$

Eliminating a and b from (2) and (3), we get:

$$\boxed{F(x, y, z, p, q) = 0}$$

This is required P.D.E.

Problem :- Find P.D.E. by eliminating h and k by equation $(x-h)^2 + (y-k)^2 + z^2 = c^2$

Solve :- Given that

$$(x-h)^2 + (y-k)^2 + z^2 = c^2 \quad \text{--- (1)}$$

Differentiate (1) Partially w.r.t. x, we get;

$$2(x-h) + \cancel{2(y-k)^2} + 2z\frac{\partial z}{\partial x} = 0$$

$$(x-h) \cancel{+ (y-k)} = -zp \quad \text{--- (2)}$$

Similarly, differentiate (1) partially w.r.t. y, we get;

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$$(y-k) = -zq \quad \text{--- (3)}$$

Now, $①^2 + ②^2$:-

$$(x-h)^2 + (y-k)^2 = z^2 (p^2+q^2)$$

Using ①

$$c^2 - z^2 = z^2 (p^2+q^2)$$

$$z^2 + z^2 (p^2+q^2) = c^2$$

$$\boxed{z^2 (1+p^2+q^2) = c^2}$$

This is required P.D.E..

Problem-② Obtain P.D.E.

$$z = (x+a)(y+b)$$

Solve :- Given that

$$z = (x+a)(y+b) \quad -①$$

Differentiate ① w.r.t. x, we get;

$$\frac{\partial z}{\partial x} = (y+b)\cancel{(a)}$$

$$p = (y+b) \quad -②$$

Similarly, differentiate ① w.r.t. y, we get;

$$\frac{\partial z}{\partial y} = (x+a)$$

$$q = (x+a) \quad -③$$

Using ② & ③, equation ① becomes

$$\boxed{z = pq}$$

This is required P.D.E.

Problem-③ Form a P.D.E. $x^2+y^2+(z-c)^2 = a^2$

Solve :- Given that

$$x^2+y^2+(z-c)^2 = a^2 \quad -①$$

Differentiate ① w.r.t. x, we get;

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x = -(z-c)p \quad -②$$

Similarly, differentiate ① w.r.t. y, we get;

$$2y + 2(z-c)^2 \frac{\partial z}{\partial y} = 0$$

$$y = -(z-c)^2 q \quad -③$$

~~Now, ①^2 + ③^2; we get;~~

$$x^2 + y^2 = (z-c)^2 pq$$

~~Using ①,~~ $x^2+y^2 = a^2 - (z-c)^2$

Now, ② ÷ ③; we get;

$$\frac{x}{y} = \frac{p}{q}$$

$$\boxed{py - qx = 0}$$

This is required P.D.E..

Problem - ③ Find P.D.E.

$$z = Ae^{pt} \sin px$$

where A and p are constants.

Solue:- Given that

$$z = Ae^{pt} \sin px \quad -①$$

Differentiate ① w.r.t. x, we get;

$$\frac{\partial z}{\partial x} = Ap.e^{pt} \cos px \quad -②$$

Similarly, differentiate ① w.r.t. t, we get;

$$\frac{\partial z}{\partial t} = Ape^{pt} \sin px \quad -③$$

Again differentiate ① w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = -Ap^2 e^{pt} \sin px$$

and, $\frac{\partial^2 z}{\partial y^2} = Ap^2 e^{pt} \sin px$

Adding, we get,

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0}$$

This is 2nd order P.D.E..

Exercise :-

Find the P.D.E.

1. $z = a(x+y) + b$
2. $z = ax + a^2y^2 + b$
3. $az + b = a^2x + y$
4. $ax^2 + by^2 + z^2 = 1$
5. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solve :- ① $z = a(x+y) + b$

Given that $z = a(x+y) + b$ — ①

Differentiate partially ① w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = a$$

$$p = a \quad \text{— ②}$$

Differentiate partially ① w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = a$$

$$q = a \quad \text{— ③}$$

from ② & ③ :-

$$\boxed{p = q}$$

2) $z = ax + a^2y^2 + b$

Given that $z = ax + a^2y^2 + b$ — ①

Differentiate partially ① w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = a + 0 + 0 = a \quad \text{— ②}$$

Differentiate partially ② w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = 0 + 2a^2y + 0 = 2a^2y \quad \text{— ③}$$

$② ^3 \div ③$

$$\frac{\left(\frac{\partial z}{\partial x}\right)^2}{\frac{\partial z}{\partial y}} = \frac{\cancel{a^2}}{2\cancel{a^2}y}$$

$$\frac{p^2}{q} = \frac{1}{2y}$$

$$\boxed{2yp^2 = q}$$

3) $az + b = a^2x + y$

Given that $az + b = a^2x + y$ — ①

Differentiate partially ① w.r.t. x, we get,

$$a\frac{\partial z}{\partial x} = a^2 \quad \text{— ②}$$

$$\frac{\partial z}{\partial x} = a \quad \text{— ②}$$

Differentiate partially ② w.r.t. y, we get,

$$a\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{a} \quad \text{— ③}$$

② × ③

$$\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = \cancel{a} \times \frac{1}{\cancel{a}}$$

$$\boxed{pq = 1}$$

4) $ax^2+by^2+z^2=1$

Given that $ax^2+by^2+z^2=1$ —— ①

Differentiate partially ① w.r.t. x, we get,

$$a \times 2x + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$2ax + 2z\frac{\partial z}{\partial x} = 0$$

$$ax = -z\frac{\partial z}{\partial x} \quad \text{—— ②}$$

Differentiate partially ① w.r.t. y, we get,

$$2by + 2z\frac{\partial z}{\partial y} = 0$$

$$by = -z\frac{\partial z}{\partial y} \quad \text{—— ③}$$

from ② & ③, we get,

$$a = -\frac{pz}{x} \quad \& \quad b = -\frac{qz}{y}$$

$$-\frac{pz}{x} \times x^2 + \frac{-qz}{y} \times y^2 + z^2 = 1$$

$$\boxed{-pzx - qzy + z^2 = 1}$$

5) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Given that $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ —— ①

Differentiate partially ① w.r.t. x, we get,

$$\frac{2\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{a^2} \quad \text{—— ②}$$

Differentiate partially ① w.r.t. y, we get

$$\frac{2\,\partial z}{\partial y} = \frac{2y}{b^2} \longrightarrow$$

$$\frac{\partial z}{\partial y} = \frac{y}{b^2} \quad \text{—— ③}$$

from ② & ③, we get,

$$a^2 = \frac{x}{p} \quad \& \quad b^2 = \frac{y}{q}$$

putting them in ①, we get,

$$\frac{x^2}{x} \times p + \frac{y^2}{y} \times q = 2z$$

$$\boxed{2z = xp + qy}$$

⑥ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— ①

Differentiating partially ① w.r.t x, we get,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\frac{x}{a^2} = -\frac{z}{c^2}\frac{\partial z}{\partial x}$$

$$p = \frac{-xc^2}{a^2 z} \quad \text{—— ②}$$

Differentiating partially ① w.r.t. y, we get,

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$$

$$\frac{y}{b^2} = \frac{-z}{c^2}\frac{\partial z}{\partial y}$$

$$q = \frac{-yc^2}{b^2 z} \quad \text{—— ③}$$

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from ② & ③, we get,

$$a^2 = \frac{-xc^2}{pz} \quad \& \quad b^2 = \frac{-yc^2}{qz}$$

putting in ①,

$$\frac{x^2}{-x\cancel{c^2}/pz} + \frac{y^2}{-y\cancel{c^2}/qz} + \frac{z^2}{-z\cancel{c^2}/rz} = 0$$

$$\boxed{-xpz - yqz - zrz = 0}$$

Elimination of arbitrary function :-

Problem :- from the P.D.E. $z = f(x^2 - y^2)$

Solve :- Given that $z = f(x^2-y^2)$ —①

Diff. partially ①, w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\, 2x$$

$$p = f'(x^2-y^2)\, 2x \quad \text{—②}$$

Diff. partially ① w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$$

$$q = f'(x^2-y^2)(-2y) \quad \text{—③}$$

Now, ② ÷ ③

$$\frac{p}{q} = \frac{\cancel{2}x}{-\cancel{2}y}$$

$$\boxed{py + qx = 0}$$

Problem :- from the P.D.E. of $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Solve :- Given that $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Diff. partially ①, w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right) \times \left(\frac{-1}{x^2}\right)$$

$$x^2 p = -2f'\left(\frac{1}{x} + \log y\right) \quad \text{———} \ (2)$$

Similarly, diff. partially (1) w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\left(\frac{1}{y}\right)$$

$$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{———} \ (3)$$

(2) ÷ (3)

$$\frac{px^2}{qy - 2y^2} = \frac{-2f'\left(\frac{1}{x} + \log y\right)}{2f'\left(\frac{1}{x} + \log y\right)}$$

$$px^2 = 2y^2 - qy$$

$$\boxed{px^2 + qy = 2y^2}$$

This is required P.D.E.

Problem (3):- Find the P.D.E. of $z = f(x+iy) + g(x-iy)$

Solve:- Given that $z = f(x+iy) + g(x-iy)$

Diff. partially (1) w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy)$$

Again, diff. partially w.r.t. x

$$\frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy) \quad \text{———} \ (2)$$

Now, diff. partially w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = if'(x+iy) - ig''(x-iy)$$

Again,

$$\frac{\partial^2 z}{\partial y^2} = -f''(x+iy) - g''(x-iy) \quad \text{—} \ (3)$$

Adding ② & ③, we get,

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0}$$

This is required P.D.E.

Formula ÷ Let $u(x, y, z)$ and $v(x, y, z)$ are function of x, y, z connected by relation

$$f(u, v) = 0 \quad \text{——} ①$$

then corresponding partial differential equation

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\boxed{Pp + Qq = R}$$

where $P = \dfrac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix}$

$Q = \dfrac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix}$

$R = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

Problem ÷ Find the P.D.E. $f(x+y+z, x^2+y^2+z^2)$

Solve ÷ Let, $u = x+y+z$

$v = x^2+y^2+z^2$

the corresponding P.D.E. is

$$Pp + Qq = R$$

Now, $P = \dfrac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix}$$

$$= 2(z-y)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial x} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2z & 2x \end{vmatrix}$$

$$= 2(x-z)$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix}$$

$$= 2(y-x)$$

then ① becomes :-

$$2(z-y)p + 2(x-z)q = 2(y-x)$$

$$\boxed{(z-y)p + (x-z)q = (y-x)}$$

This is required P.D.E.

Solution of partial differential equation :-

Let, $f(x, y, z, p, q) = 0$ —— ① be given partial differential equation

then solution of ① is a form

$$\boxed{F(x, y, z, a, b) = 0}$$

Types of Solution :-

① Complete Integral — The PDE in which the number of arbitrary constant is equal to order of PDE is called complete solution or complete integral.

② Particular Integral — The PI is one of the solution of complete integral.

Lagrange linear partial differential equation :-

The lagrange linear PDE is of the form

$$\boxed{Pp + Qq = R} \quad \text{—— ①}$$

where P, Q, R are function of x, y, z

Solve :- **Step-①.**

Lagrange auxiliary equation is

$$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R}$$

(I) (II) (III)

Step-②.

Solve (I) and (II), (I) and (III)

we get,

$$u(x, y, z) = c_1$$

$$v(x, y, z) = c_2$$

Step - 3.

Solution is $f(u,v)=0$ or $\phi(u,v)=0$

OR

$v=\phi(u)$

$u=\phi(v)$

Problem :- $p\tan x + q\tan y = \tan z$

Solu :- The lagrange auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

I II III

Consider I & II

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\int \cot x\,dx = \int \cot y\,dy + \text{Constant}$$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log \frac{\sin x}{\sin y} = \log c_1$$

$$\boxed{\frac{\sin x}{\sin y} = c_1} \quad \text{--- (1)}$$

Now, from II & III

$$\frac{\partial y}{\tan y} = \frac{\partial z}{\tan z}$$

Integrating we get,

$$\int \cot y\,dy = \int \cot z\,dz + \text{Constant}$$

$$\log \frac{\sin y}{\sin z} = \log c_2$$

$$\boxed{\frac{\sin y}{\sin z} = c_2} \quad \text{--- (2)}$$

from ① and ②, the required solution is

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right)=0$$

OR

$$\boxed{\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right)}$$

Problem :- $x^2p + y^2q = z^2$

Solve :- Lagrange auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

(I) (II) (III)

Consider (I) & (II)

$$\frac{\partial x}{x^2} = \frac{\partial y}{y^2}$$

Integrating,

$$\int x^{-2}dx = \int y^{-2}dy + \text{Constant}$$

$$\frac{-1}{x} = \frac{-1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

$$\boxed{\frac{x-y}{xy} = c_1}$$

Consider (II) & (III)

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

Integrating

$$\frac{-1}{y} = \frac{-1}{z} + c_2$$

$$c_2 = \frac{1}{z} - \frac{1}{y}$$

$$\boxed{c_2 = \frac{y-z}{yz}}$$

$$\boxed{\phi\left(\frac{x-y}{xy}, \frac{y-z}{yz}\right) = 0}$$

Lagrange Multiplier Method :-

$$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R} = \frac{\partial x + \partial y + \partial z}{P+Q+R}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{xP + yQ + zR} = \frac{dx \pm dy}{P \pm Q}$$

Problem :- $(y-z)p + (z-x)q = (x-y)$.

Solve :- The lagrange auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

(I) (II) (III) (IV)

$$\frac{x\,dx + y\,dy + z\,dz}{0} \quad \text{— (V)}$$

from (I) and (IV)

$$dx + dy + dz = 0$$

Integrate

$$\int dx + \int dy + \int dz = c_1$$

$$x + y + z = c_1 \quad \text{— (1)}$$

from (I) and (V)

$$x\,dx + y\,dy + z\,dz = 0.$$

Integrating

$$\frac{x^2+y^2+z^2}{2} = \text{Constant} = c_2$$

from (1) & (2), the required solution is

$$\boxed{f(x+y+z,\ x^2+y^2+z^2) = 0}$$

Problem ③ :- $x(y-z)p + y(z-x)q = z(x-y)$

Solve :- The lagrange auxiliary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{0} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

I II III IV V

from (IV) & (I)

$$dx + dy + dz = 0$$

Integrate

$$x + y + z = c_1 \quad \text{---} \ (1)$$

from (V) & (I)

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrate

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2$$

$$xyz = c_2$$

from (1) & (2)

the required solution is

$$\boxed{f(x+y+z,\ xyz) = 0}$$

OR

$$\boxed{x+y+z = f(xyz)}$$

Problem :- $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Solve :- The lagrange auxiliary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{y^2-xy} = \frac{dx+dy+dz}{x^2+y^2+z^2-(xy+yz+zx)}$$

I II III IV

$$= \frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dx-dy}{(x-y)(x+y+z)} \quad \text{V}$$

$$= \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

VI VII

From (IV) & (V)

$$\frac{dx-dy}{(x-y)\cancel{(x+y+z)}} = \frac{dy-dz}{(y-z)\cancel{(x+y+z)}}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

Integrate

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$\frac{x-y}{y-z} = c_1 \quad \text{--- (1)}$$

Similarly by (V) & (VI)

$$\frac{y-z}{z-x} = c_2 \quad \text{--- (2)}$$

From (1) & (2)

The required solution is

$$\boxed{f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0}$$